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ABSTRACT

This study observed and documented the changes of college students' notion of mathematics functions in a virtual environment that integrated a synchronous and asynchronous Computer Mediated Communication system. Four Arizona State University undergraduates were teamed into pairs and solved function problems in the virtual classroom over a one month period. A protocol analysis method was used to analyze the collected data on students' conceptual change of function concept. The study found that most of the students limited conceptions and misconceptions were changed. Students' conceptual change did not follow a linear pattern, and the change did not occur easily, especially when the notion of functions was well established. Some of the students' limited conceptions and misconceptions of functions were interrelated. The notion that a function problem should make sense played an important role in the development of students' function concept. Finally, students corrected most of their incorrect conceptions regarding the notion of functions. This means that college students could learn through interactions in this type of virtual learning environment. (Contains 40 references.) (Author/ASK)

A Case Study on the Changes of University Students' Function Concept in a Virtual Environment

by

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ABSTRACT

This study observed and documented the changes of college student's notion of mathematics functions in a virtual environment that integrated a synchronous and asynchronous Computer Mediated Communication system. Four Arizona State University undergraduates were teamed into pairs and solved function problems in the virtual classroom over one month period. A protocol analysis method was used to analyze the collected data on student's conceptual change of function concept. The study found that most of the students limited conceptions and misconceptions were changed. Student's conceptual change did not follow a linear pattern, and the change did not occur easily especially when the notion of functions was well established. Some of the students limited conceptions and misconceptions of functions were interrelated. A notion, a function problem should make sense, played an important role in the development of students function concept. Finally, students corrected most of their incorrect conceptions regarding the notion of functions. This means that college students could learn through interactions in this type of virtual learning environment.

The Need to Investigate the Cognitive Process of Learning Function Concept

The notion of function has been recognized as the fundamental and unifying principle of mathematics (Brieske, 1973). Most mathematicians and teachers of secondary and post-secondary mathematics would agree that functions are an important concept in learning mathematics (Selden & Selden, 1992).

However, several research findings (for example, Even, 1990, 1993) have revealed that misconceptions or limited notions about functions are common among students. Moreover, Leinhardt, Zaslavsky, and Stein (1990) concluded from an extensive review of studies of functions that few studies have investigated elementary and secondary level student's cognitive processes relating to the development of function concept. Little is known about the cognitive process for secondary and post-secondary level students. The cognitive process of function concept acquisition, however, is an essential factor that needs to be integrated into the course outline (Selden & Selden, 1992). Therefore, more research is needed regarding student learning of function concept (Even, 1989; Fischbein, 1990).

Studying student's function misconceptions or limited notions could help us understand the developmental process of function concept because student's function misconceptions and limited understanding of function conceptions are closely related to the cognitive process of the student's learning of functions. Several studies (Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Sfard, 1992; Schwingendorf, Hawks, & Beineke, 1992) have found that student's misconceptions and limited function notions changed as their knowledge of functions progress. It was also found that function misconceptions and limited function notions have prevented many students from acquiring a broader and deeper understanding of functions (Dreyfus & Eisenberg, 1984;

Wagner, 1981). Therefore, understanding of the cognitive process of learning functions need to be accompanied by investigations into conceptual changes in student's function misconceptions and limited notions. Furthermore, investigating and understanding the changing process of students misconceptions is beneficial for developing curriculum materials, teaching strategies and instructional activities, as well as providing new opportunities for understanding the way students learn (Driver, 1981). Vinner (1983) supported this idea by stating that revealing student's concept image, which reflects student's misconceptions and limited notions, provides teachers with a better understanding of students and suggestions for improving instruction.

Leinhardt et al. (1990), Even (1993) and one current textbook (Larson, Hostetler, & Edwards, 1996) defined a function as a "relation with a rule," that contains two essential features, "univalence" and "arbitrariness." Based on this modern concept of function, the rule of relation (or correspondence), is the main notion of function determining the characteristics of the relation between the elements of two sets. The function rule is applied to find a value of a variable y corresponding to a given value of variable x. More concrete examples of the function rule (or the rule of correspondence) can be a function formula expressed in algebraic form (e.g., $y = 2x + 2$) or a function graph since the graph itself is acquired by applying the function rule to each value of x. The range of a function cannot be defined without applying the rule of function to the domain (Orton, 1971). (i.e., the correspondence of elements in the domain to elements in the range depends on the rule of correspondence.) (Ayers, Davis, Dubinsky, & Lewin, 1988).

The rule of correspondence contains other aspects. Vinner (1983) argued that the notion of function relation should not be limited to the characteristic of the "rule" since the rule contradicts the arbitrary aspect of functions. Therefore, the major components of Dirichlets' function definition, the concepts of "arbitrariness" and "univalence," should be considered in relation to the rule of correspondence (Even, 1989, 1990, & 1993). The arbitrary nature of functions is about "the relationship between the two sets in which function is defined and the sets themselves" (p. 96). Moreover, the notion of "arbitrariness" excludes the perspective that only a certain expression with a specific set of elements can define functions. The notion of "univalence" requires that each element of the domain in the function is to correspond to one, and only one, element of the range of the function. Selden and Selden (1992) described Dirichlet's definition of a function as "a special kind of correspondence between two sets" (p. 2). It is an arbitrary correspondence between variables "so that to any value of the independent variable, there is associated one and only one value of the dependent variable" (Ponte, 1992, p. 4).

These statements reveal the various aspects related to function relation and imply the necessity of acquiring the appropriate notion of "relation" to correctly understand functions. Describing the concept of relationship as a tool to describe and predict change, Sierpinska (1992) argued that functions should be taught as models of relationship in the first place emphasizing the importance of understanding functions as a concept of relationship. Sierpinska viewed the notion of function as "a result of the human endeavor to come to terms with changes observed and experienced in the surrounding world" (p. 31). Referring to the variables x and y as the "world of changes" or "world of changing objects," Sierpinska described the symbol *f* as the "world of relationships between changes or changing objects or the world of processes that transform objects into other objects" (p. 31). Based on these arguments, it is clear that the function relationship (or the function rule) is the central notion to be acquired in learning functions, even though several other concepts related to functions such as domain, range, variables, etc. should not be ignored.

Several studies (for example, Vinner & Dreyfus, 1989) also identified the misconceptions and limited notion related to the rule of correspondence. The case study reported in this article, therefore, investigates the changes in the limited conception and misconceptions of the rule of correspondence to provide a better understanding on how university student's function concept develops.

<u>Cognitive Process of Learning Concepts and Constructivists' View on Learning</u>

Constructiviests view a student's cognitive process as an adaptive act of organization so their experiences "fit" with previously constructed knowledge (Davis, Maher & Noddings, 1990). Knowledge develops and continues to change as learners interact with their learning environments (Brown, Collins, & Duguid, 1989). Many researchers in mathematical education take a constructivist perspective on knowledge acquisition as they investigate students' thinking and learning processes (Selden & Selden, 1992). Several studies of function concept development applied instructional treatments which were based on constructivist theory of learning in their studies. Sfard (1992) and Schwingendorf et al. (1992) applied some constructivist learning conditions, such as exposing students to several types of function representations, leading open discussions about the concepts of functions, or applying a cooperative learning (or problem solving) environment to the teaching of functions. These studies found that the constructivists learning conditions helped students understanding of functions. Carlson (1996) and Dubinsky and Harel, (1992) also concluded that students' understanding of functions was improved as a result of engaging students in constructive activities.

<u>Interactions and cooperative group learning.</u> Cooperative group learning format has been widely used by constructivists. This makes sense because the major aspect of group learning, interactions, is one of the main conditions for constructivist learning environments. It has also been argued that learning a mathematical concept can be enriched through group learning because students can reach a deeper level of mathematics understanding through communicating with each other in small group (Johnson, 1983). Student's learning through interactions in a group has also been endorsed by professional recommendations such as NCTM (1989, 1991) and Mathematical Association of America Committee on the Mathematics Education of Teachers (1991). There have been quite a number of studies (for example, Webb, 1982, 1991) that have investigated the influence of small group learning methods on college level mathematical performance. These studies reported that applying the cooperative group learning method resulted in positive effects on student's mathematics achievement, especially for improving mathematical concept acquisition (Dees, 1991; Shaughnessy, 1977).

The Trends of Internet in Education and its Role in Providing Interactive Cooperative Group Learning in Mathematics.

A clear trend of applying internet technology to classroom exists. The U.S. Department of Education reported 35% of public schools have access to the internet, and in 1996 an additional 14% have access to other wide-area networks such as CompuServe, America Online, and Prodigy (Follansbee, Gilsdorf, Stahl, Dunfey, Cohen, Pisha, & Hughes, 1996). This trend has been encouraged by educators, especially those in the area of distance-learning (McIsaac & Gunawardena, 1996) as well as government leaders.

With increased access to the internet, applying on-line communication to collaborative group learning is growing. This is because asynchronous or synchronous Computer Mediated Communication [CMC] technology has various features that can simulate group learning environments. In fact, the effect of CMC for group learning in distance education was investigated in some studies (Kaye, 1990; McConnell, 1990). The studies concluded that CMC has a potential to be a useful tool for collaborative group learning environments. Davie and Wells (1991) also claimed that design of a collaborative learning as a part of a CMC classroom will empower students by establishing a community of learners who contribute to a group effort. The potential of CMC as a group learning condition was also explored in other studies (Schwartz & Froehlke, 1991; Hansen, Shong, Kubota, & Hubbard, 1991; Knuth & Goodrum, 1991). They agreed that CMC has various advantageous features for group learning. Some of their findings were that CMC (a) allowed students to consult with others about their work; (b) created a learning community where faculty and students can integrate their outside classroom experiences into the electronic leaning environment; (c) gave students equal opportunity to participate in group discussions; (d) enabled students to work and interact cooperatively with each other at times and places convenient to them; and (e) enhanced students' critical thinking while working in a group.

Considering the potential benefits of a CMC system for group learning, and the popularity and availability of on-line communications, investigating group learning via a synchronous and asynchronous CMC system can contribute to discovering a better application of CMC to the mathematics education. Furthermore, considering the possible benefits and the learning condition of the CMC system where students depend totally on writing to communicate with others, it is necessary to briefly review the related benefits for learning a concept through writing. First of all, writing has been described as "meaning-making processes that involve the learner in actively building connections between what he is learning and what is already known" (Mayher, Lester, & Pradl, 1983, P. 78). Writing helps students to (a) comprehend and internalize subject content (Flores & Mendoza, 1996); (b) explore their knowledge about a topic; (c) become aware of their own thought processes (Marwine, 1989; Powell & Lopez, 1989); (d) facilitate their understanding of conceptual relationships (Abel & Abel, 1988; Pearce & Davison, 1988); (e) facilitate "personal ownership" of knowledge (Connolly, 1989; Mett, 1989); (f) contrast their positions at different points in time (Flores & Mendoza, 1996); (g) clarify their own ideas or opinions (Gribbin, 1991); and (h) think and organize their thoughts (Abel & Abel, 1988; Flores & Mendoza, 1996).

Especially when we know that the current mathematics education cannot afford to exercise a small group discussion during the class mainly because there are too many things to teach (Carlson, 1996; Selden & Selden, 1992), investigating the potential of CMC system for mathematics learning is worth because the synchronous and asynchronous CMC enable college students to have more freedom in terms of the time or place constraint so that they can be more affordable in participating after class mathematics group learning activity via a CMC.

Method

Study Purpose

The purpose of this study was to observe, analyze, and document the changes in college student misconceptions or limited conceptions regarding the notion of function relation in a virtual problem solving environment. This case study was conducted to answer the following questions: Will a student's misconceptions and limited conceptions related to function relation change vertically and horizontally, if they work in a pair in a Multi-user Object Oriented (MOO) with web conferencing boards, a synchronous and asynchronous CMC environment? If a change occurs, how will the change progress? Two misconceptions and four limited conceptions were investigated. The misconceptions were: not accepting constant functions, and functions with many-to-one correspondence. The limited conceptions were: not accepting functions that have split domains, a finite number of exceptional points, an arbitrary correspondence, and no causal relationship.

The virtual environment applied in the problem solving sessions were MOO and web conferencing boards. In traditional mathematics function classes, the assignment designed for enriching concept attainment is often completed in independent problem solving activity. However, those assignments can be more beneficial if students have a chance to discuss and solve the problems in a group, because students can reach a deeper level of mathematics understanding through communicating with each other in small group (Lo, Wheatley, & Smith, 1994; Johnson, 1983; Pappas, Kiefer, & Levstik, 1990; and Lappan & Schram, 1989). The virtual environment with synchronous and asynchronous feature can be beneficial for students' group learning. The potential features of a synchronous and asynchronous CMC system for learning concept may include (a) having people focus on the message, not the messenger; (b) giving plenty of time for reflection, analysis, and composition; (c) encouraging thinking and retrospective analysis; (d) having the whole discussion available as a transcript; (e) encouraging active involvement; (f) not being constrained by geography or time; (g) providing chances to collaborate with global experts online and to access global archival resources. Therefore, it is necessary to investigate the potential of the virtual environment for mathematics concept learning. Moreover, the MOO can simulate the face-to-face problem solving condition helping the participant sense the social presence. Web conferencing board function was added to the MOO environment to provide another way to reflect their ideas and interact among group members.

Subjects

During the final exam in the spring semester of 1998 at Arizona State University, recruiting handouts were distributed to 10 different college algebra classes. A total of 12 students volunteered to participate in the study. Each volunteer was given a paper-pencil based pretest. With the test result, the researcher had an interview with the volunteer via a MOO system which simulates a virtual environment. Based on the analysis of the test and interview, four of the volunteers, two males and two females, were selected as subjects because they exhibited one or more misconceptions or limited conceptions of function relation in one or more of the three settings (e.g., algebraic formulas, graphs, and tables). They had not taken any advanced function classes.

Data Collection Procedures

Data collection occurred from May to July of 1998. Written data used in this study were obtained from the paper/pencil based pretest and posttest, and the six MOO problem solving sessions. Interaction data were obtained from the MOO interviews after the pretest and posttest, discussions during the six MOO problem solving sessions, and the messages posted on the web conferencing board.

Data Collection from the Pretest and the Posttest

A paper-pencil based test was developed to identify the two misconceptions and the four limited conceptions. The same test problems were used in the pretest and posttest. The pretest was given before the six MOO problem solving sessions and the posttest was given after. There was an interval of about eight weeks between the pretest and the posttest. Both tests had a total of 20 questions. Each of the first 18 problems consisted of two parts. The first part asked whether the given relation was a function or not: (a) If the answer was yes, then the student was asked to justify the answer; and (b) if the answer was no, then the student was asked to justify the answer and to rewrite the situation as a function. Each misconception or limited conception was asked in three different representations: algebraic formulas, graphs, and tables. The 19^{th} question asked students to select all descriptions that correctly define functions. The 20^{th} question asked students to define function in their own words. Student's answers on the test were used as the reference for the following interview.

Data Collection from Two Interviews

After the pretest and the posttest, the students were interviewed by the researcher in the MOO system. The guidelines provided by Ginsburg (1981), who outlined clinical task-based interviews for psychological research on mathematical learning, was the basis for the interview design. The tests and interviews worked together to identify the misconceptions or the limited conceptions of each student.

Data Collection from Six MOO Problem Solving Sessions

The primary purpose of the MOO problem solving sessions was to enhance the students' learning by providing students with experiences in solving function problems through interactions in a virtual classroom. Five constructivist learning conditions, suggested by Driscoll (1994), that enhance interactions between students and their learning environments were modified to be applied in the design of the problem solving sessions. Each problem solving session was designed to:

- experience conflicts with their function concept when they solve problems or have discussions with their partner.
- solve the problem cooperatively exchanging opinions, criticizing each other's ideas and solutions positively, and cooperating to reach a conclusion or a solution.
- experience the same conception using three types of representations: graphic, algebraic formula, and table.
- revisit the problems they solved and reviewed the solutions they reached so that the students can analyze, criticize, and compare them with the solutions to the present problems.
- give the initiative to the students for all the activities of each session while the role of the researcher was to assist them.

A MOO classroom with a feature of asynchronous web conferencing boards was used to create a virtual environment. The MOO classroom resided in a virtual University called "Diversity University (Diversity University, http://du.org / telnet moo.du.org 8888)." The classroom contained several couches, a blackboard, a teacher's desk, recorders, a wall clock, etc. Those objects were virtual objects. The MOO classroom was a text-based environment. This means that students interacted with each other through written text only. Students could interact with each other synchronously or asynchronously in the classroom, but most of the interactions occurred in real time. A couch was assigned to each pair and students had to sit on the couch to exchange opinions while solving the problems with their partner. The main reason for this was to localize each pair's conversations in the classroom. If a student needs to say something to all the participants, he or she could speak up. The conversations and interactions of each pair were recorded by an MOO recorder which was placed on each couch.

Students used web conferencing boards during each problem solving session. The web conferencing board contained four links. The first link connected to a page where students could connect to the virtual classroom. The page connected by the first link also had directions on how to connect to the classroom using a telnet program. The second link connected to a page that contained some essential MOO commands. Each MOO command was explained with detailed descriptions and examples. The next link gave general guidelines on how to behave in the virtual classroom. The last link was connected to interactive web pages where the 36 function problems were listed.

During each session, students used the interactive web conferencing boards to review their answers for the old problems as well as others exchanging comments asynchronously. The comments exchanged through the web conferencing board were collected as well. The interactive web conferencing boards consisted of 36 web pages and each web page contains one function problem with a feature for a student to interact asynchronously with the other students. In case of emergency, they used a telephone to inform the researcher, then the researcher helped them to solve the problem.

Each pair had six problem solving sessions. Each session consisted of six function problems, one problem for each of the two misconceptions and the four limited conceptions. Three types of representations (i.e., graphs, tables, and algebraic formulas) were used in each session. The six problem solving sessions had the same format. Problem solving sessions were spaced three to four days apart. During the problem solving sessions, they hardly felt the need to use a graph, table, or an equation since all they need to do was discussing whether the problems were a function or not. When they had to refer the graph or table during their problem solving sessions, they used words to describe a certain part, for example, of a graph.

Data Analysis Procedures

Adapting Dreyfus and Eisenberg's Model

For design of this study, I modified the open-ended three dimensional function block model developed by Dreyfus and Eisenberg (1982, 1984). They used it for the systematic assessment of students' intuitions on the mathematical notion of functions. The model considers functions as a three-dimensional block: (a) Its x-axis represents function settings; (b) its y-axis represents notions of functions; and (c) its z-axis represents levels of abstraction of each notion. In my study, the investigation of function concept change focused on only two dimensions, vertical (z-axis) and horizontal (x-axis). The level of y-axis for this study was fixed to one notion, focusing only on the notion of function relation. The levels of z-axis (vertical change) were also limited to the four limited conceptions and the two misconceptions regarding the notion of function relation. As the number of misconceptions or the limited conceptions decreased, the higher the level of abstraction or generalization on the axis of the student's notion of function relation. The levels of x-axis (horizontal change) were also limited to three function settings: graph, table, and algebraic formula.

Protocol Analysis for the Notion of Function Relation

The protocols for the notion of function relation were collected from four parts of the study: pretest and interview; pair problem solving sessions one to three; pair problem solving sessions four to six; and posttest and interview. Each part contained 18 problems that were designed to investigate two misconceptions and four limited conceptions that related to the notion of function relation in three different function settings. The 18 problems of each part handled the six notions in three different representations. This means that there was only one problem that dealt with one of the six notions using one of three representations in each part. So, one of the 18 problems from one part can be matched to one of the 18 problems from the other part in terms of the conception they dealt with and the representation they used. However, the problems were not identical in content.

The researcher analyzed the protocols by reading them in four different ways. First, the transcripts of all of the interviews and pair problem solving sessions were arranged based on the type of problem being discussed. Thus, 18 groups of scripts were prepared. After reading and analyzing the scripts, the researcher rearranged them into six groups according to the two misconceptions and four limited conceptions the problems were designed to detect. The researcher reread the transcripts within each group together, trying to determine the existence of those misconceptions and limited conceptions. Once each student's misconceptions or limited conceptions were identified, the transcripts were regrouped by individual student and examined. Finally, the researcher conducted a spot reading of the transcripts to find similarities among students' statements or ideas. For example, if student A's statements for causal relation in graphic setting were similar to the explanations for arbitrary correspondence from student C, they were examined together.

The focus of the protocol analysis was identifying and describing the differences at the level of the two misconceptions and the four limited conceptions on the notion of function relation. A coding scheme developed by the researcher was used to identify and describe differences at each level. Using the coding scheme, the degrees of existence of the misconceptions and the limited conceptions were sorted into three categories: strong evidence of existence, strong evidence of non-existence, and some evidence of existence. Examples of correct explanations and incorrect explanations for each problem in the pretest and the six sessions were also made in advance by the researcher. The coding scheme and examples were kept in mind and used as a referential rubric for interpreting the protocols.

Analysis for vertical change (z-axis). Protocols of each student's problems solving activity from the four parts (pretest, first three pair problem solving sessions, second three pair problem solving session, and posttest) were analyzed to document the change of the misconception or the limited conception in each setting. Any progress (or deterioration) in the level of existence of a misconception or limited conception across the four parts was observed and described to document the change in the misconception or limited conception. The vertical change in the notion of function relation was documented based on the observation of the change in the six conceptions.

Analysis for horizontal change (x-axis). Protocols of each students' problem solving activity from the four parts were analyzed to document the horizontal change of the misconceptions and the limited conceptions across the three settings: algebraic formulas, graphs, and tables. In each part, the horizontal progress was documented by observing each misconception and limited conception across the three settings.

Results

In this section, the four subjects are described individually first. After the general descriptions of each subject, each subject's vertical and horizontal change of misconceptions and limited conceptions are summarized and reported. The overall overview on the conceptual change of all students will be introduced in the "Discussions and Implications" section.

A General Description of Each Student

Student A. Student A was a 23 years old female who had taken intermediate and college algebra. She was double Spanish and International Business Management major and had computer experience with Microsoft Excel and Corel Word Perfect, as well as some e-mail and Netscape. Her limited conceptions were function with causal relation, split domains, and exceptional values. She was not enthusiastic about solving function problems via the MOO and she preferred to work face to face. She, however, felt comfortable communicating with a partner and optimistic about working in a MOO environment. She expected that the problem solving activities would help her understand functions better.

Student C. Student C was a 18 year old male student who had taken intermediate algebra and college algebra. His major was business. He described himself as computer literate and seemed quite comfortable using e-mail and Netscape. He had difficulty with two misconceptions (many-to-one correspondence, and constant functions) and a limited conception (causal relation). Student C was enthusiastic about solving function problems in MOO. He indicated a preference to solving problems in a MOO rather than face to face since he had time to think the problems over in a MOO. He felt confidence explaining in the MOO environment. Moreover, he was comfortable working with others in the MOO. He also expected that the problem solving activities in the MOO environment would help him understand functions better.

Student R. Student R was a 19 year old male student who had taken intermediate algebra and college algebra. His major was physics and used a computer for school work. Student R showed his limited conception on causal relation in three settings. Other than this limited conception, he did not show any other limited conceptions or misconceptions regarding the notion of function relation. He thought solving function problems in a MOO would enhance his knowledge of functions. He was optimistic about solving problems in a MOO although he preferred solving problems face to face. He was comfortable working in a MOO while enjoying exchanging ideas with others. He explained that the MOO environment relieved tensions since he did not see his partner.

Student T. Student T was a 23 year old female who had taken intermediate algebra and college algebra. Her major was family resources and human development. She indicated a low level of computer literacy. Student T had a limited conception of arbitrary correspondence and causal relation. She felt that solving the function problems in a MOO would help her to understand function better. She, however, would prefer to solve the mathematics problems face to face rather than in a MOO. Although she was not confident about working in a MOO, she was comfortable communicating with her partner in a MOO.

The Descriptions on the Conceptual Change of Student A

Vertical change of causal relation. There was a vertical change in her limited conception of causal relation in all three settings and the change showed a similar pattern: Her limited conception became concrete as the study progressed, then she changed her limited conception losing her confidence on the requirement of causal relation in a function. In the pretest problem 19, she chose 'a' which stated that "function relation must have some kind of causal relationship between the element of two set." In the posttest problem 19, on the contrary, she chose 'b', "a function can have a causal relation between the elements, but it is not a necessary condition." However, she was not fully confident on her choice of the posttest problem 19. Her definition of function in the posttest problem 20 also described her inconclusive position at the end of the study:

Student A: Passes the VLT has one x value only per y value, and maybe, but I am not sure, has real-world application. Forms a line that doesn't have any places where it is vertical and maybe has to make common, real-world sense.

Vertical change of split domains. In pretest, she did not show the limited conception of a function graph with split domains. Rejecting the graph with split domains in session one, however, she clearly showed her limited conception of split domains. Furthermore, she mentioned that she had never seen a piecewise function before. She also showed her incorrect understanding of the graph. Her limited conception for this graph, however, was not due to the split domains of the graph but to her incorrect understanding of the graph, a function graph can not have a disconnected part. In session five, she improved her understanding of disconnected function graph by accepting it as a function. Based on the fact that the disconnected part of a graph is an important aspect of a function graph with split domains, it can be concluded that there was a vertical change in her limited conception of function graphs with split domains.

Vertical change of finite number of exceptional points. In the pretest, student A showed her difficulty in accepting graphs with exceptional points as functions. Her difficulty, however, was stemmed from a disconnected part of the graph which was created by an exceptional value. Once she understood the meaning of the discontinuous part in a graph in session five, however, she did not show any further difficulty in interpreting a graph with exceptional points. Her answer for the posttest problem with an exceptional value and her answer for the posttest problem 19 demonstrated her correct understanding of function graphs with exceptional values, for both connected and disconnected graphs. Therefore, there was a vertical change in her limited conception of the function graph with exceptional points.

Horizontal change on causal relation. Throughout the study, student A did not show the horizontal change in her limited conception of causal relation among the three settings. Showing her ability to identify the lack of the causality in all three pretest problems, she consistently argued the necessity of causal relation between values. Once the sessions started, she did not make any mistake in determining the existence of causality in the problems without showing any variance on her understanding of the causality issue among the three settings. When she changed her position regarding the causal relation in session five table problem, she showed the same position that she showed for the algebraic problem in session six. During the posttest, she also demonstrated the same undecided position toward the causal relation in all the three settings. Therefore, there was no horizontal change in her limited conception of causal relation among the three settings.

Horizontal change on split domains. In the pretest, student A did not show any limited conception of split domains in the three settings. She, however, showed her limited conception for a function graph with split domains in session one. The limited conception was mainly due to her incorrect understanding of a function graph, that a function graph must have a continuous pattern. Arguing that a function graph could not be a function if any of its parts were disconnected from the rest, she rejected the graph with split domains as a function. She stated that she did not really care about the pattern of the graph as long as the graph was continuously connected. This was the same argument that she made for the disconnected function graph with exceptional points in the pretest and session three. As the study progressed, however, she improved her understanding of a disconnected function graph. In session three and the posttest, she accepted the disconnected function with an exceptional value as a function. She also considered the constant function graph with split domains showing a disconnected pattern as a function graph in session five. Considering that the discontinuous pattern of the graph is an important attribute of a function graph with split domains, her rejection of the disconnected graph as a function can be considered as a limited conception of split domains in graphic setting. In conclusion, student A's understanding of the function problem with split domains in graphic setting was different from the two settings until the beginning of session three, showing the limited conception only for problems with the disconnected graph. This means that there was a horizontal change on her limited conception of split domain until session three.

Horizontal change on finite number of exceptional points. Student A's answers in the pretest indicated that she had a limited conception of exceptional values only in a graphic setting. For the pretest graphic problem, she answered that a function graph should not be disconnected and rejected the function graph with an exceptional value detached from the main body of the graph as a function. She supported this position, a function graph should be continuous, in the pretest problem 19 by choosing an answer, "function relations can be defined with exceptional points." This answer, however, was accompanied with a graph containing a continuous pattern with exceptional values. She revealed the same difficulty with the graph in session three rejecting the graph as a function because it was disconnected. This rejection was due to her belief that a function graph must be represented with a continuous line. In fact, she made the same argument for the disconnected graph in session one problem one, the graph was not a function because it was disconnected. After she understood that the disconnected line can be a function graph with her partner's help in session three, she did not show the difficulty with the discontinuous graphs in the following sessions. With the problem in table and algebraic settings, the student did not demonstrate a limited notion on finite number of exceptional values throughout the study. In conclusion, her understanding of the function problem with exceptional values in graphic setting was different from the two settings from the pretest until session three showing her limited conception only for problems with the disconnected graph.

The Descriptions on the Conceptual Change of Student C

Vertical change of causal relation. His conception of causal relation was vertically changed in all three settings showing a similar pattern. At first, he did not show any limited conception of causal relation. In the pretest problem 19, he also chose b, a causal relation between variables was not a necessary condition for functions. Until session five, he kept this position firmly in all three settings. In session five, however, he was convinced that a function needs a causality after having a discussion with his second partner, student R. So, he changed his position agreeing that a real-life function must have a causal relation in it otherwise it would be dealing with a nonsense. His understanding of causality in function remained this way until session six although there was a reversion of his original position in session five (problem six) and a sense of uncertainty in session six (problem four). In the posttest, he went back to his original position because he did not have any confidence on the assertion that a function must have a causal relation. He showed the change of his understanding in the three posttest problems in the three settings. His position on causality was reaffirmed in the posttest problem 19 choosing b as a correct description of a function.

Vertical change of constant functions. His misconception of constant functions changed vertically in all three settings. In the pretest, he showed the misconception in all three settings. Choosing g in problem 19, "if more than one element of the domain matches with the same element of the range, it is not function anymore," he defined a function as "a relationship where any x value corresponds to only one y value and vice versa." This reaffirmed his misconception. He also showed his difficulty in determining the inputs and outputs of a constant function graph in pretest and session two. While showing his difficulty in interpreting the algebraic problem in the pretest, his answer for session one problem implied his misconception of constant functions in algebraic formula setting. For the rest of the problems in all three setting, however, he demonstrated his correct conception of constant functions. In the posttest problem 19, he chose h, "In a function, all elements of the domain could be matched with one same element of the range." He also defined a function as "When only x input matches to only one y output. Two inputs can share one output, just not the other way around."

Vertical change of many-to-one correspondence. Student C showed the misconception of many-to-one correspondence in algebraic and table setting in the pretest. With the problem in graphic setting, he gave the correct answer without showing the misconception. His correct answer for the graphic setting, however, contradicted his answer in the pretest problem 19 and his definition of functions in the problem 20. In problem 19, he chose g while rejecting f and h as a correct description of functions. This indicated that he would accept only one-to-one correspondence as a condition for a function and this was consistent with his answer for the pretest problems in the table and the algebraic formula setting. He reaffirmed this argument in his definition of function, an input corresponds to only one output and vice versa. He also did not accept the horizontal line graph in the pretest as a function because it violated the function rule, one-to-one correspondence.

Once the session started, he did not show this misconception again except in session two algebraic problems. He, however, showed a difficulty in identifying the variables correctly with the graphic problem in session three and four. For the posttest problem 19, he demonstrated his correct understanding of many-to-one correspondence by choosing f and h. His definition of function in the posttest also showed that he understood the function correspondence as univalence (i.e., one or many values of the domain matches with only one value of the range). In conclusion, student C's limited conception of many-to-one correspondence was changed vertically in algebraic formula and table setting. However, his conception in graphic setting did not show a vertical change since he did not show the limited conception throughout the study period. His understanding of a graph with many-to-one correspondence, however, improved.

<u>Horizontal change on causal relation.</u> In the pretest, student C showed the following position toward the causal relation in all the three settings: Causality was not a necessary condition for deciding the eligibility of functions. His answer for the pretest problem 19 reaffirmed this argument. He gave this same response to the causality issue until session four in all three settings. While he was revisiting the session four graphic problem in session five, however, he began to think about the causal relation in functions. Then he decided to change his answer concluding that the graph in session four was not a function because it did not have a causal relation. The main reason why he changed his opinion regarding causal relation in function was the followings: (a) He did not have much opportunity to think about causality issue, and (b) he agreed with his partner's assertion that a function needs to make a sense so that it can represent a sensible real-life situation. This conclusion was based on his conversation with his partner in session five. However, in the immediately following problem (six) in session five, he still argued that the causal relation in the table was not an important factor in determining a function. Then, he agreed with his partner showing the limited conception on causal relation with an algebraic equation in session six (problem four).

In the posttest, he decided to keep his original position regarding the causality. The causality should not be required for a function, because he was not convinced of the necessity of causal relation in functions. His answer for the posttest problem 19 confirmed this position. Although he kept his original position on the causal relation issue, there was a difference between his original understanding on the causality issue in function and his conception on the causal relation at the end of the study. In other words, his original position was stemmed from not having opportunity to think about causality and functions. However, his position in the posttest, which still showed the lack of confidence on whether or not a function needs to have causal relation in it, was the result of his understanding on the issue after he had the opportunity to talk and think about it. In conclusion, student C did not show any difference in his understanding of causal issue among the three settings until session four. In session five and six, however, he showed different understanding on causality. While he showed the limited conception of causal relation with the graphic problem in session five and with the algebraic problem in session six, he gave the correct answer for the table setting problem in session five. There was no definitive evidence that the difference was due to his different understanding of representations. In fact, his consistent position toward causality issue in the pretest, session one to four, and in the posttest was a strong evidence that his notion of causal relation was not influenced by the representations of problems.

<u>Horizontal change on constant functions.</u> The change of his understanding on constant functions from the pretest to session three can be described in two different stages, stage of misconception (pretest) and stage of establishment (session one to three). During the pretest, student C showed his misconception of constant functions in all the

three settings. His misconception was based on his belief on the one-to-one correspondence (i.e., each value of the domain matches with only one value of the range, and vice versa). In fact, this was the way he defined a function in the pretest problem 20. Therefore, he did not show any difference in his misconception of constant functions among the three settings during the pretest.

In session one (problem six), student C gave a contradictory answer showing his unstable understanding on the constant function with algebraic equation. He said that it was a function because each day corresponded with the same height. He contradicted this answer with his argument that two inputs could not share the same output. He, however, showed his improved understanding by considering the pattern of constant functions as unimportant factor. For the constant function graph in session two (problem six), student C interpreted the value on the horizontal axis as output and the value on the vertical axis as input. This led him to conclude that one input matched with many output making it non-function. This was the same mistake he made in interpreting the input and the output of a constant function graph in the pretest. In the following session, however, he was able to correct the mistake and accepted the graph as a function. This showed his improved understanding on the constant function graph. In session three (problem five), he said that the table was a function because no input shares two different outputs. He also considered the constant corresponding pattern in the table as acceptable for function. This clearly demonstrated that he did not have the misconception of constant functions in table setting.

After session three, he did not show any misconception of constant functions in any of the three settings. He correctly interpreted the algebraic equation as a constant function while demonstrating his ability to draw the constant function graph out of the algebraic equation in session four. Without making any mistake in identifying the inputs and outputs of a constant function graph in session five (problem four), he accepted the horizontal line as a function graph. In session six (problem six), student C explained that several inputs can share the same output but not the other way around. The table was a function because each input (month) matched to only one output (5 trillion). In the posttest, he did not show any misconception of constant functions, and did not show any variance in his understanding of constant functions among the three settings. In conclusion, there was no difference in his misconception of constant functions among the three settings during the pretest. In session one and two, he showed his unstable understanding with the algebraic and graphic setting problem of constant functions. From session three, however, his conception of constant functions was not varied among the three settings.

<u>Horizontal change of many-to-one correspondence.</u> He clearly showed that he had the misconception of many-to-one correspondence for the problems with the table and algebraic setting during the pretest. With the graphic problem one in the pretest, however, he did not show the misconception. In session one (problem five) with table setting, he began to realize that many-to-one correspondence could happen when each input has only one output. He noticed that the output remained the same, but he said that it was all right because the input changed so that each input had only one output making it a function. He, however, again showed the misconception of many-to-one

correspondence when he attempted to solve the algebraic equation in session two problem five. He answered that the equation was not a function because some x values had the same y value. Then, he accepted the equation as a function agreeing with his partner's argument, the time (input) was changing constantly but the output value can remain the same. After session two, he did not show the misconception of many-to-one correspondence in all the three settings. In session three problem six, however, he showed his difficulty with identifying the inputs and outputs of the graph. The misinterpretation of the graph led him to conclude that it was not a function because the input at some point corresponded to multiple outputs. He made the same mistake when he revisited this problem in session four. Although he had incorrect understanding on the graph with a horizontal line, he did not show the misconception of many-to-one correspondence because he answered that the graph in session three was not a function because some inputs had multiple outputs. He also stated that the horizontal pattern of graph was not an important factor to consider in determining a function. In session six and the posttest, he correctly identified the input and output of the graph with many-to-one corresponding pattern, while demonstrating correct understandings on the graph with many-to-one correspondence. In conclusion, there was a horizontal change in his conception of many-to-one correspondence during the pretest, session one, and session two.

<u>The Descriptions on the Conceptual Change of Student R</u>

<u>Vertical change of causal relation.</u> His limited conception of causal relation changed vertically in all three settings. In the pretest, he demonstrated the limited conception in all three settings. He reaffirmed his limited conception in the pretest problem 19. He, however, had an unstable understanding of causality revealing his difficulty in defining a causality in the pretest algebraic problem, the session one problem in graphic setting, and the session two problem in table setting. As the study progressed, his limited conception became firm and sophisticated while making a series of decisions regarding the causality in functions: (a) categorizing functions into two groups, "pure mathematical" and "real life," (b) applying the requirement of causality only to real-life problems, and (c) considering the condition of causality in the first place when solving the problem with real-life situations. He demonstrated this position again in the posttest, defining a function as followings:

> Student R: A function is a relation which, in mathematical sense, passes the VLT & has only one output for each individual input. In a real-life situation, a function is a relation which shows dependence of the dependent variable or the independent variable. In other words, the independent variable somehow causes or affects the dependent variable.

Although, student R's limited notion was not corrected, his unstable position on this notion became stable one, it can be concluded that there was a vertical change on his notion of causal relation.

<u>Horizontal change on causal relation.</u> His limited conception of causal relation did not show much difference across the three settings. In the pretest, student R

demonstrated the same limited conception of causal relation in all three settings, that a causal relation must exist in any function problem with a real-world situation. However, he showed that his understanding of causal relationship was not firm with the pretest algebraic problem. He could not decide whether the algebraic problem was a function or not because the situation given to the problem did not make a "real-world" sense. In session one, graphic problem four, he also showed his incorrect understanding on the causality of the problem. Student R initially thought that the graph was a function because there was a causal relation between the inputs and outputs. He said that they were causally related since where (output) the grasshopper landed depended on what time (input) it jumped. Then he realized that the inputs and outputs of the graph were based on random activity, so he became confused. However, his experience with function problems assured him that time was a function for everything because time is related to all real-life events. Therefore, he concluded that the graph was a function because there was a causal relation between the time the grasshopper jumped and the place it landed.

In session two, he had a similar problem in determining causal relation for the table. At first, student R accepted the table as a function because each x value had only one y value. He was also convinced that there was a causal relationship between the inputs and outputs of the table but he could not prove it since it required additional information (the weight of Joe, the trajectory of the dart) and skills (knowledge on statistics and probability). One of the main reasons for his belief on the existence of causality in the table was the regular corresponding pattern in the data of the table. The possibility, that someone blind-folded could get better in throwing darts over time, further convinced him that there could be a causal relation in it. Based on these, he concluded that the problem was a function. These incidences showed that he had difficulty in determining whether or not the situations in these problems contained a causality. This implied that his understanding of causal relationship were not firmly established. After all, he showed the difficulty in identifying a causality in a problem in all three settings but the difficulty had nothing to do with the settings of the problems.

After solving session two problem, he did not have any further trouble in determining the existence of causal relation in a problem. He also did not hesitate in deciding whether the problem was a function or not based on the existence of causality. He further rationalized the necessity of causal relation in functions while his understanding of causality in functions became more concrete. For example, he categorized the types of function into two, functions with a purely mathematical situation and functions with a real-life situation. Furthermore, the causal relation was a prerequisite only for function problems with real-life situations. He reaffirmed this in his definition of function in the posttest. In conclusion, he did not show any difference on his limited conception of causal relation during the whole study period across the three settings.

The Descriptions on the Conceptual Change of Student T

Vertical change of causal relation. In all three settings, her limited conception of causal relation changed vertically. She showed her limited conception in the pretest in all three settings. Her answer for the pretest problem 19, "function relations must have some kind of causal relationship between the elements of two sets," confirmed her limited conception. Throughout all six sessions, she did not demonstrate the limited conception except for the session five problem in table setting. At the beginning of session five, she rejected the table as a function arguing the necessity of causal relation in the table. Later of the session, she changed her position. Another problem regarding causality was shown in the graphic setting problems in session four and the posttest. For these two problems, she used causality in the problem situation to determine the input and output values of the graph. However, she did not insist the limited conception of causal relation for the problems. In the posttest, her answer for problem 19 and her definition of functions reaffirmed that she did not have the limited conception as well. In problem 19, she answered that "a function can have a causal relation between the elements, but it is not a necessary condition to be a function." She also defined a function as "a function does not have to have a causal relationship, as long as it works mathematically and does pass the requirements... there can be one and only one input value for each output value. This input value can never be repeated or duplicated. However, the output values can remain constant..." Therefore, it can be concluded that her limited conception of causal relation changed vertically in all three settings after the pretest.

However, she made an important remark regarding the development of her conception of causal relation in session five. Right after she reversed her argument, a function needs a causal relation, she said that she disregarded the causality in the previous sessions without understanding the reason behind. This implied that her correction of the limited conception after the pretest was not based on her firm understanding of causality in function but on routine practice. It was not clear, however, what was her understanding regarding the causal relation at session five since she simply said that she understood the reason.

Vertical change of arbitrary correspondence. Her limited conception of arbitrary correspondence changed vertically in graphic and algebraic formula setting. She showed her limited conception in the pretest graphic and algebraic formula problems. She reaffirmed the limited conception by choosing c in the pretest problem 19, "if a relation between the input elements and output elements does not show regularity or some predictable patterns, then the relation is not a function." She added that e could not be a correct description of function because "an arbitrary correspondence cannot have any causal relation." This showed not only her limited conception of arbitrary correspondence but also that the limited conception was related to her limited conception of causal relation. After the pretest, she did not demonstrate the limited conception anymore in both settings. In the posttest problem 19, she confirmed her correct understanding of arbitrary correspondence in functions by choosing e as a correct description of a function. In conclusion, there was a vertical change in her limited conception of arbitrary correspondence in graphic and algebraic formula setting.

Horizontal change of causal relation. In the pretest, student T's limited conception of causal relation appeared in all the three settings. Her answer for the pretest problem 19 confirmed her limited conception of causal relation. After the pretest, her main concern was the vertical line test paying little attention to causality when she classified a function. In session four problem five, however, she used the causal relation in the problem situation to determine the input and output value of the graph. She also temporarily showed the limited conception of causal relation in the beginning of session five (problem six) rejecting the table as a function because there was no causal relationship in the data of the table. When the researcher suggested her to consider whether a mathematician would care about reality when working on mathematical ideas, she changed her answer accepting the table as a function. She said that she had disregarded causal relation in functions in the previous sessions without having any clear understanding the reason why a function did not need a causal relationship but now she understood the reason why it was not necessary to consider the causality issue. This means that she did not have a firm understanding of causality issue and functions. This can be the reason why she made a sudden change in her position regarding the causality issue after the pretest without insisting on her opinion with her first partner who simply ignored the issue. Throughout the study, however, that was the only time she temporarily showed her limited conception of causal relation. In conclusion, her understanding of causal relationship did not vary among the three settings during the whole period of the study.

Horizontal change of arbitrary correspondence. In the pretest, student T showed her limited conception of arbitrary correspondence in the algebraic and graphic setting. Student T did not classify a graph (problem sixteen) with arbitrary correspondence as a function because of the irregular correspondence between the input and output values. She expected the shape of the graph to be "smooth" and predictable. She also suggested to fix the graph into a regular bell shape. Classifying the algebraic equation (problem twelve) as a non-function, student T also argued that the equation should have a regular corresponding pattern in it. She, however, did not show the limited conception in the problem with the table setting. She rejected the table as a function because it violated the function rule, one y value for each x value. She also provided a correct solution to make the problem a function, taking out one of the x values which was repeated. The fact -- she gave this suggestion despite the description of the problem clearly stated that there was no regular corresponding patterns between the x and y values -- is contradicting to her answers for the algebraic equation and the irregular function graph. In sessions one to six, she demonstrated that she did not have a limited conception of the arbitrary correspondence in all three settings. In the posttest, she continuously disregarded the issue of irregular correspondence pattern. In conclusion, she showed the difference in her understanding of the arbitrary correspondence among the three settings in the pretest. During the rest of the study, however, she did not have the limited conception at all, showing no sign of horizontal change among the three settings.

Discussions and Implications

Student A

The case of student A showed how difficult it is to change a well-established conception. With her first partner, student A was able to transform her limited conception of causal relation to a more sophisticated, but incorrect, conception which she continued to maintain while working with her second partner. After session five, student A became unsure whether a function must have a causal relation or not, giving inconclusive answers to all the causal relation posttest problems. Her definition of function regarding causal relation was unclear as well. This is an indication that her conceptual development on causal relation reached a new phase where she had to re-evaluate what she knew and seek a new explanation or meaning of functions.

Regarding student A's limited conception of split domains, she demonstrated this only in one problem in a graphic setting. Her limited conception was related to her mistake in interpreting a graph. Students making mistakes in interpreting graphs were also what several other studies noted (for example, Carlson, 1996). Most of the difficulties found in these studies were due to unnecessarily giving meaning to the shape of graphs or incorrectly interpreting the values on x and y axes. Similarly, student A interpreted the linear part of a graph with split domains in session one ($y = -ax + b$) to indicate that foreign travelers were spending a "negative amount of money." She also considered negative x values of the graph as "negative time."

Student A had an incorrect image of a function graph: She thought that all the values of a function graph should form one continuous line or curve. This tendency among students to favor continuous function graphs over disconnected ones was found in other studies. For example, Kerslake (1981) found that students tend to connect discrete points in a graph when it is inappropriate to do so. Supporting this finding, Markovits et al. (1986) found that half of the ninth graders who were asked to identify an algebraic function such as $f(x) = 4x + 6$, with the domain and range restricted to natural numbers in a graphic setting, accepted the graph with a continuous line as the correct one. Most of the subjects did not accept the graph with the appropriate discrete points as a function. College students and junior high school teachers also showed adherence to continuity in graphs (Vinner & Dreyfus, 1989; Even, 1993). Here, student A's belief that a function graph must be continuous was a direct cause of her limited conception of split domains in graphic setting. She rejected the graph as a function simply because it was disconnected.

Student A rejected graphs with exceptional points as functions in the pretest and in session three because the graphs were disconnected at those exceptional values. An exceptional value in a graph, she argued, could not be a value since the line of the graph did not go through it. The finding--that her limited conceptions of function graphs with split domains and function graphs with exceptional points were both due to her incorrect image of a function graph--is an example of how a student's limited conceptions may be interrelated. Since the limited conceptions had the same cause, they both disappeared as student A corrected her image of a function graph during the rest of the sessions.

<u>Student C</u>

Student C's misconceptions of many-to-one correspondence and constant functions need to be considered together, because he gave the same reason for rejecting them as functions: An x value (input) must correspond to only one y value (output) and vice versa. During the pretest, student C showed this misconception with many-to-one correspondence problems in table and algebraic settings, and with constant function problems in all three settings. His definition of function in the pretest confirmed his misconception of one-to-one correspondence. This incorrect understanding of univalence was also found in other studies (Marnyanskii, 1969; Thomas, 1975; Vinner, 1983).

Vinner (1983) suggested that the reason for this misconception might derive from the student's belief that if each input matches with only one output, then the reverse has to be true. Likewise, Even (1993) discovered that few prospective secondary mathematics teachers had a firm understanding of the role of univalence in learning functions.

Student C's misconception of one-to-one correspondence gradually disappeared as the MOO sessions progressed although the process of correcting his misconception was not consistent. In session one, he accepted a table with many-to-one correspondence as a function. With the algebraic constant function in the same session, however, he gave a contradictory answer: The equation was a function because each input had the same output, but two inputs cannot share the same output in functions. With the algebraic problem of many-to-one correspondence in session two, he at first indicated his misconception by arguing that inputs cannot share the same output. Then, with his partner's help, he corrected his answer, accepting the equation as a function. Student C did not show this misconception again during the rest of sessions.

This way of changing a concept, by following an inconsistent process, was also found when he changed his limited conception of causal relation. Although Student C's understanding of causality seemed to be firm until session five problem five, he changed his position after he and his second partner had a serious discussion regarding causal relation and its application in real life. His partner's argument (that a function problem with a real-life situation needs to have causality or else the problem could not make sense) appealed to his common sense. Student C agreed that it would be meaningless if a function had to deal with nonsense. However, student C was reluctant to apply what he had decided from that discussion to the subsequent problem in the same session. During the next session, with an algebraic formula problem, student C was not absolutely sure about the necessity of a causal relation in a real-life problem although he agreed with his partner that the problem needed to have a causal relation. This showed that his understanding toward causality was contradictory. During the posttest, student C returned to his original position regarding causal relation in functions. He explained that he did not have confidence in his partner's argument.

There are two noticeable aspects in the development of student C's notion of causal relation. First, his second partner's argument was quite influential. The argument was rooted in a sound reasoning appealing to common sense and it temporarily changed student C's firm understanding of causal relation in functions. The second aspect is the resilience of student C's notion of causal relation. Student C did not easily discard what he had leaned from his function class despite his partner's persuasive argument. This indicates that a well-established understanding may not be changed easily even by receiving one or two contrary but reasonable explanations.

Student R

The vertical change of student R's limited conception of causal relation showed a similar pattern in each of the three settings: His incomplete limited conception of causal relation became firm and concrete as he solved more problems. Originally, while his limited conception was being established, he showed a common difficulty in all three settings: a difficulty in determining a causal relation in problems that contained a nonsensical real-life situation. Eventually, he became able to determine these problems as non-functions in all three settings. Student R further developed his understanding of causal relation, classifying function problems into two types, "purely mathematical" problems and real-life situations, and he applied causality to problems with real-life situations only. He also rationalized why "pure" mathematical problems did not need a causal relation.

The main rationale for student R's belief on the necessity of causality in functions was "sense-making": A real-life function problem must make sense. While he was solving the pretest problems that did not have causality, he came to realize the importance of causal relations for a real-life function problem since the problem did not make sense without them. During the study, he frequently emphasized that a real-life function problem has to make sense since the purpose of learning mathematics, he thought, was to apply its theoretical method to real-world problems. Therefore, it would be meaningless to teach functions that address nonsense. By making sense out of a problem, student R tried to connect his understanding of functions to real-world situations. His rationale was a convincing argument and his first partner, student A, was so convinced that she didn't overcome this limited conception until the end of the study. His second partner, student C, who had strongly argued that a function had nothing to do with causality, also became temporarily convinced that a function with real-life conditions must have a causal relation.

Student R made an interesting comment as to why he previously had not noticed the need for causality in real-life function problems. He stated that all previous real-life situation function problems in his function class used actual data from the "real world" which made sense. Most real-life function problems in textbooks use sensible conditions which would produce reasonable data patterns. Student R argued that he became convinced that all function problems that related to real-life situations must make sense after having experience with only sensible textbook problems. This is consistent with what Leinhardt et al. (1990) pointed out, students' insufficient experience with function problems dealing with exceptional conditions is one of the reasons for their incomplete understanding of functions.

Another reason why student R was convinced that functions must have a causal relation was his understanding of the terms "function" and "dependent variable." He said, "To say something is a function of something else is basically to say that it is a direct result of a particular action. Hence, the name 'dependent variable.'" This means that the function-related term and its connotation, interdependence, reaffirmed his belief on causality in functions. Just as student R developed his limited conception of causal relation from his notion of dependency, "A depends on B," other studies (Marnyanskii, 1969; Vinner, 1983; Vinner & Dreyfus, 1989) reported a similar finding. These studies found that students transformed the concept of dependency into the concept of causal relation while defining a function as "a dependent relation between two variables." Reviewing related studies, Leinhardt, Zaslavsky, and Stein (1990) concluded that students might have developed the idea that the concept of dependency is the same as the concept of causal connection. Based on this finding, it can be recommended that the terms (e.g., dependent variables, functions) or the instructional context (e.g., the value y is determined by the value x in an algebraic function formula) which could contain the connotation of dependency must be carefully used in a function class.

Student T

In the pretest, student T stated her belief that the inputs and outputs of functions should be causally related to each other. Immediately after the pretest, however, she ignored causal relation issue. This drastic change in her position towards the causal relation of functions can be understood by her admission in session five that she was not sure about the reason why she did not care about causal relation. This clearly indicates that she disregarded the issue after the pretest without knowing the reason. By simply ignoring the causal relation issue like her first partner (student C), she easily corrected the limited conception of causal relation that she had in the pretest. This suggests that her correct understanding of causal relation after the pretest was not a properly established notion.

Student T's unstable notion of causality can also be seen from her temporary agreement with her second partner during session five that the real-world problem needed a causal relation to be a function. After considering the researcher's suggestion to think about the relation between mathematical ideas (or concepts) and real-life conditions, she came to realize why it was not necessary to consider causality for functions. She demonstrated her understanding of causal relation in session six by confirming that the situations given to a function problem should not be considered in determining whether it was a function or not. This demonstration of her conception of causal relation was based on the acquired belief that mathematical ideas should not be influenced by real-world situations.

As with her limited conception of causal relationship, student T indicated a limited conception of arbitrary correspondence in the pretest, but did not demonstrate this afterwards. Although it was not clear why she suddenly disregarded arbitrariness after the pretest, her argument in the pretest, a function must have a "nice" corresponding pattern and function graphs need to have a "smooth" and predictable pattern, has also been identified in several other studies. Even (1993) found that prospective secondary mathematics teachers expected "niceness" and "smoothness" from function graphs. Tenth- and eleventh-graders who participated in a study by Vinner (1983) stated that a function graph must show a regular and reasonable pattern. The students excluded arbitrary correspondence from the rule of functions. College students and high school teachers also defined a function as a corresponding rule matching inputs and outputs (Vinner & Dreyfus, 1989) while they expected a "certain regularity" from a function (Sfard, 1992; Vinner & Dreyfus, 1989).

Student T's pretest answers also indicated that her limited conception of arbitrariness was related to her limited conception of causal relation. She stated that a function with an irregular correspondence could not be a function because "arbitrary correspondence cannot have any causal relation." It would be natural for student T to consider that the conceptions of arbitrary correspondence and causal relation could not coexist because the notion of causal relation is usually associated with "reasonableness," "predictability," or "regularity," which cannot be found in arbitrariness. Although this is an assumption, this might be the reason why student T realized that arbitrariness in functions was acceptable during session one, once she and her first partner decided to ignore causal relation as a necessary condition for a problem to be a function. Furthermore, an abrupt change in her position on causal relation and arbitrary correspondence was noticed in session one. Therefore, her notion of arbitrary correspondence in functions needs to be understood as related to her limited conception of causal relation.

Conclusions

The analysis of student interactions, explanations and interviews revealed that the notion of sense-making played an important role in students' understanding of functions. All students were influenced by the notion of sense-making in some way when they solved function problems that lacked causality. This sense-making process often became combined with other limited conceptions or misconceptions, thus creating a major obstacle to students' understanding of functions. For example, one student rejected a function graph as representing a function since the graph displayed nonsensical data. This conclusion indicated the students' insufficient skill in interpreting a function graph and a sense-making rationale combined, leading the student to an incorrect conclusion. These findings indicate that it is important to understand how the notion of "sense-making" is related to student function concept development.

The data also revealed that three of the students' limited conceptions and misconceptions were interrelated. For example, the limited conception of split domains was related to the incorrect image of a function graph and led to the rejection of a graph with exceptional values. A misconception of many-to-one correspondence was closely related to the misconception of constant functions. Arbitrary correspondence was rejected as an aspect of a function due to limited conception of causal relation. Understanding the inter-relationships among the misconceptions and limited notions will, therefore, provide a deeper understanding of students' function concept development.

Another frequently observed aspect was that students' conceptual changes were not linear. They often changed their notions at one point and then returned to their original position in the following session. This indicates that conceptual change does not occur easily, especially if the original concept is well established.

The data analysis clearly showed that the learning condition used in this study at least temporarily corrected students' misconceptions and three of the four limited conceptions regarding their notion of function relation. This means that college students can learn through interactions in this type of virtual learning environment. Finally, this study indicated a potential of MOO environment as a tool to observe and collect data concerning students' conceptual changes.

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